MIDTERM: ALGEBRA II

Date: **27th February 2018**
The Total points is **110** and the maximum you can score is **100** points.

(1) (6+6+6+6+6=30) Mark all correct options.
  (a) Which of the following field extensions are algebraic?
    (i) $\mathbb{Q}(\pi/18)/\mathbb{Q}$
    (ii) $\mathbb{Q}(\mathrm{Cos}(\pi/18))/\mathbb{Q}$
    (iii) $\mathbb{C}/\mathbb{Q}$
    (iv) $\mathbb{C}/\mathbb{R}$
  (b) Which of the following numbers can be expressed in radicals?
    (i) $\mathrm{Sin}(2\pi/17)$
    (ii) Zeros of the polynomial $x^3 - 27x^2 + 2x + 2018$.
    (iii) Zeros of the polynomial $x^5 - 80x - 5$.
    (iv) Zeros of the polynomial $x^5 - 8x + 7$.
  (c) Which of the following field are the splitting field of $X^{11} - 3$?
    (i) $\mathbb{Q}(\sqrt[11]{3})$
    (ii) $\mathbb{Q}(\sqrt[11]{3}, e^{2\pi i/11})$
    (iii) $\mathbb{Q}(\sqrt[11]{3}, \mathrm{Cos}(2\pi/11))$
    (iv) $\mathbb{Q}(\sqrt[11]{3}, \mathrm{Cos}(2\pi/11), i)$
  (d) Which of the following field extensions are Galois?
    (i) $\mathbb{F}_p(x)/\mathbb{F}_p(x^p - x)$ where p is a prime, x is an indeterminate and $\mathbb{F}_p$ is the field of p elements.
    (ii) $\mathbb{Q}(\sqrt[3]{7})/\mathbb{Q}$
    (iii) $\mathbb{R}/\mathbb{Q}$
    (iv) $\mathbb{Q}(\sqrt[4]{3})/\mathbb{Q}(\sqrt{3})$
  (e) Let G be the group of automorphisms of the field $\mathbb{Q}(\sqrt[5]{2}, e^{2\pi i/5})$ and $H = \{\sigma \in G : \sigma(\sqrt[5]{2}) = \sqrt[5]{2}\}$ and $I = \{\sigma \in G : \sigma(e^{2\pi i/5}) = e^{2\pi i/5}\}$. Then which of the following is true?
    (i) $|G| = |H| \cdot |I|$.
    (ii) H and I are normal subgroups of G.
    (iii) G is generated by H and I.
    (iv) G is abelian.

(2) (5+15=20 points) When are field extensions called linearly disjoint? Let K/F and L/F be Galois extension extensions with $\mathrm{Gal}(K/F) = S_n$ for some $n \geq 5$ and $[L : F]$ is odd. Show that K/F and L/F are linearly disjoint.

(3) (8+12=20 points) Define separable field extension and normal field extension. Let K/F be a finite normal extension and $L = \{\alpha \in K : \alpha \text{ is separable over } F\}$. Show that L is a field and L/F is Galois extension.

(4) (20 points) Let F be a field and $f(x) \in F[x]$ be an irreducible separable polynomial of degree n. Let G be the Galois group of the splitting field of $f(x)$. Show that the action of G on the set of all the zeros of $f(x)$ in the splitting field is transitive. If $F = \mathbb{Q}$ and $f(x) = x^4 - 2$ then compute G.

(5) (20 points) Let p be a prime and $\mathbb{F}$ be a finite field of order p^n for some $n \geq 1$. Show that for any $a \in \mathbb{F}$, the polynomial $(X - a^p)(X - a^{p^2}) \ldots (X - a^{p^n}) \in \mathbb{F}_p[X]$.